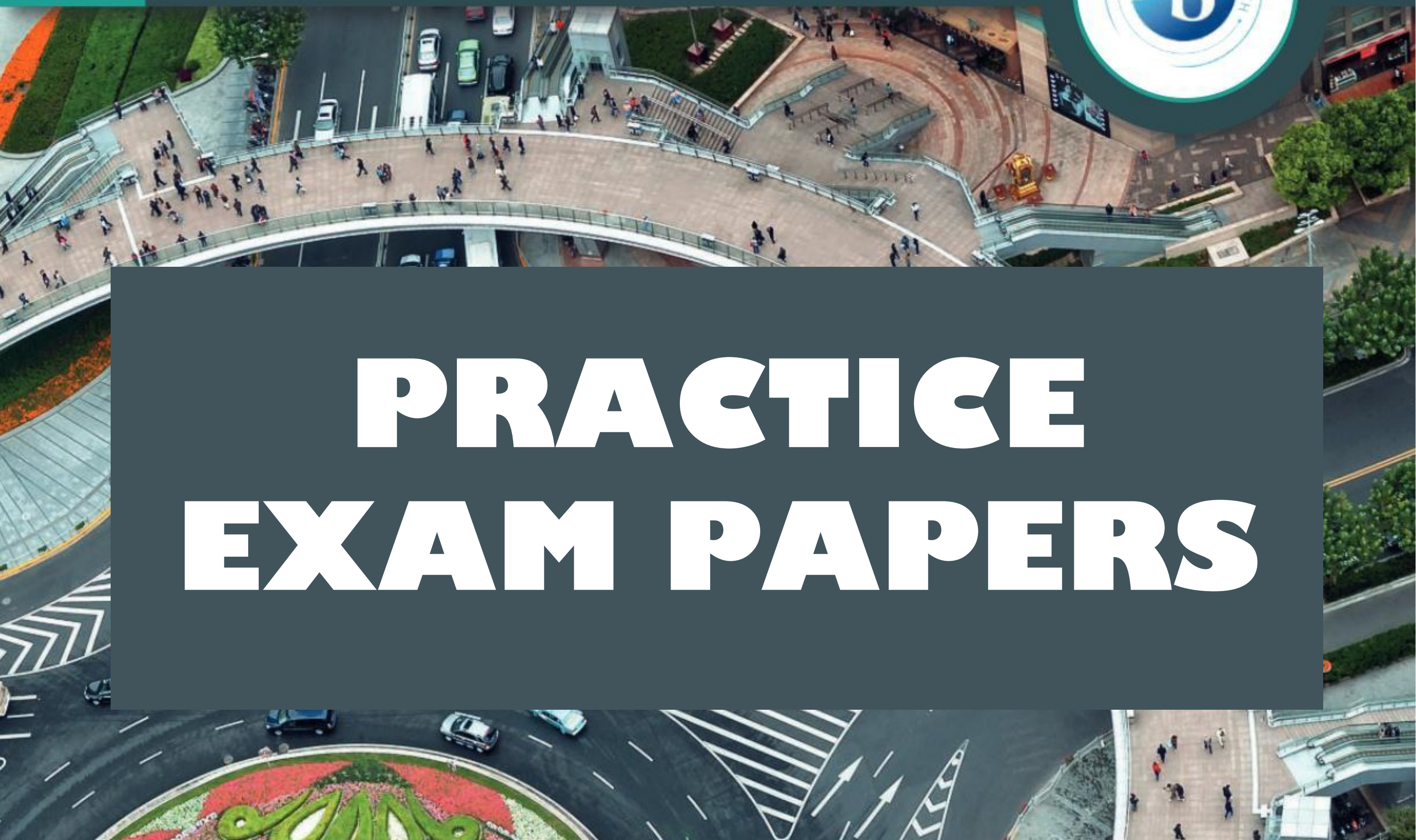
OXFORD IB DIPLOMA PROGRAMME



PRACTICE EXAM PAPERS

MATHEMATICS: APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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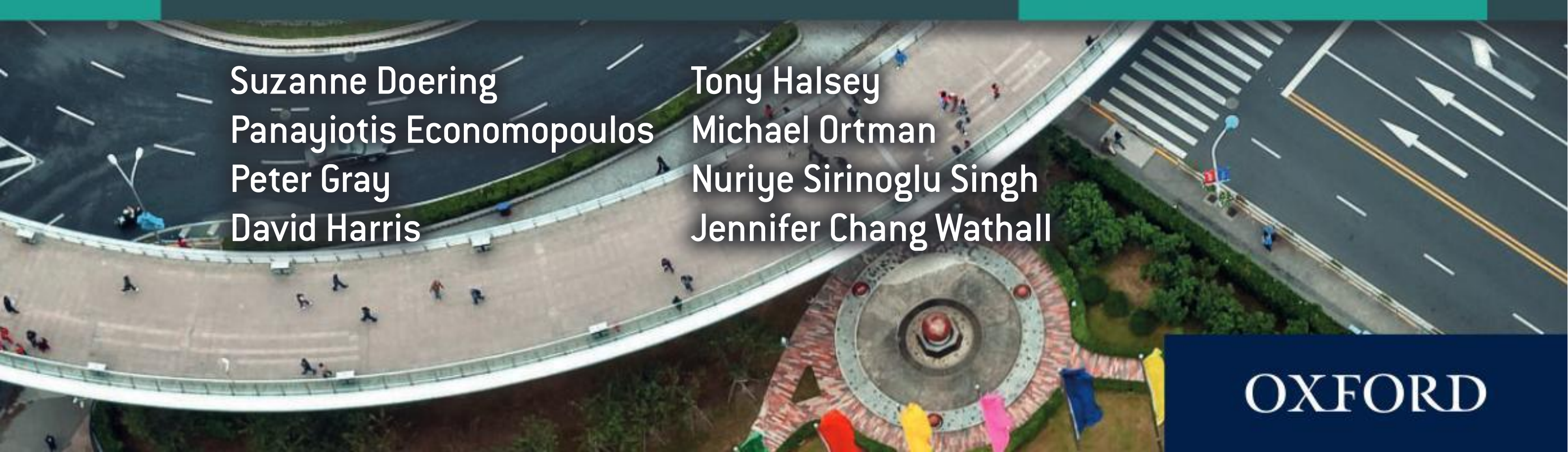


OXFORD

Paper 1

Time allowed: 2 hours

Maximum number of marks: 110 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphical display calculator for this paper.

1 [Maximum mark: 8]

Let $f(x) = 2x + 3$ and $g(x) = 4x + 5$.

a Find

i $(f \circ g)(x)$ **ii** $(g \circ f)(x)$. [4]

Let $h(x) = ax + b$ and $k(x) = cx + d$.

b Show that $(k \circ h)(x) = (h \circ k)(x)$ if and only if $d(a-1) = b(c-1)$. [4]

2 [Maximum mark: 5]

In this question give monetary answers to 2 decimal places.

Anna is going to invest money in a bank which pays 5% compound interest per year, compounded annually.

a Calculate how much Anna should deposit now, so that it will be worth £1000 in 10 years' time. [2]

Doris also invests money into an account paying 5% compound interest per year, compounded annually. She is going to deposit $£X$ in the bank initially. In 5 years' time she will take out £1000, and leave the remainder in for another 6 years. At this time, she will then take out another £1000, which will leave her account empty.

b Calculate the value of X . [3]

3 [Maximum mark: 5]

A surveyor is marking out an ornamental garden.

She starts at point A and walks in a straight line due North for 40 m to point B. She then turns clockwise and walks in a straight line for 50 m to point C. Here, she turns clockwise again and walks in a straight line for 60 m, which takes her back to point A exactly. Find the bearing of point C from point B. [5]

4 [Maximum mark: 8]

On a particular day, due to the tides, the depth of water, d metres, at the entrance to a harbour is modelled by

$d = 9 + 6\cos(30t)$,

where t is time in hours after midnight.

A yacht with a keel can only enter or exit the harbour if the depth of the water is greater than 5 m.

a Find the time intervals when the yacht can enter or exit the harbour, for $0 \le t \le 24$. Give the answers in hours to 2 decimal places. [6]

b State the period of the above function. [2]

5 [Maximum mark: 6]

Marten has four cards with the numbers $1, 2, 3, 4$ on them. Kiki has four cards with the numbers $2, 3, 4, 5$ on them. They both select one of their cards at random.

a Find the probability that both their cards have the same number. [2]

b Find the probability that the sum of the two numbers is 5. [2]

c Find the probability that Kiki's number is greater than Marten's number. [2]

6 [Maximum mark: 5]

A small firm manufactures specialist, hand-built, sports cars. Let x be the number of cars that it produces in a month and let £P be the profit that the firm makes that month. The variables are connected by the equation

$P = -10x^4 + 500x^2 - 300$ for $0 \le x \le 7$.

Note that x does **not** have to be an integer, as the firm can be part-way through completing a car at the end of a month.

a Find the profit made if 3 cars are produced. [2]

b Find the maximum profit that can be achieved in a month and the number of cars they should build in order to achieve this. [3]

7 [Maximum mark: 6]

The function $f(x)$ has the properties that $f(1) = 2$ and $\frac{df}{dx} = 3x^2 + 8x + 7$.

Find the function $f(x)$. [6]

8 [Maximum mark: 8]

The area of a sector of a circle is $\frac{9\pi}{8}$ cm^2 and the arc length of this sector is $\frac{3\pi}{4}$ cm.

a Find

i the radius of the circle

ii the angle subtended at the centre of the circle by this sector. [6]

b Find the total distance around the perimeter of this sector, giving your answer to the nearest centimetre. [2]

9 [Maximum mark: 6]

The probability distribution given below is for a discrete random variable X that represents the gain of a player in a game.

X	-2	-1	0	1	2	a
$P(X=x)$	0.2	0.1		0.2	0.1	0.01

a Find $P(X=0)$. [2]

b Given that it is a fair game, find the value of the gain a. [4]

10[Maximum mark: 6]

Consider the graph $y(x) = x^2 - 1$.

a Find the value of $a > 0$ such that $\int_0^a y(x)\,dx = 0$. [4]

b Use a sketch to give an explanation for this result. [2]

11[Maximum mark: 8]

A discrete random variable X satisfies a Poisson distribution with mean $\mu \neq 0$. It is known that

$P(X=3) = P(X=4)$.

a Find the mean of X . [3]

b Write down the variance of X . [1]

c State what can be deduced about the mode of X . [2]

This distribution models the number of pheasants that a park ranger catches on a day out. If he catches at least one pheasant then there will be complaints on social media.

d Find the probability that there are complaints made. [2]

12 [Maximum mark: 5]

Let $A = \begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}$.

a Evaluate $A^2 - 6A$. [2]

b Hence, express A^{-1} in terms of A and I. [3]

13 [Maximum mark: 8]

Let $z = x + \mathrm{i}y$. Show that if $|z + 2\mathrm{i}| = |2\mathrm{i}z - 1|$, then the point (x, y) lies on a circle. State the radius and centre of the circle. [8]

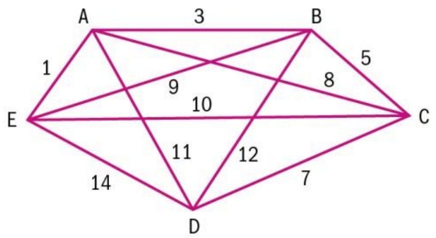
14 [Maximum mark: 8]

Gravel is falling from a chute at a rate of $2\,\mathrm{m^3\,s^{-1}}$. The gravel falls to the ground and piles up in the shape of a cone. At the vertex of the cone, the sloped surface makes an angle of $\frac{\pi}{3}$ radians with the downward vertical.

Find the rate of increase in the height of the cone when its height is $5\,\mathrm{m}$. [8]

15 [Maximum mark: 9]

A complete weighted graph is as shown below:



a Find an upper bound for the travelling salesman problem, using the nearest neighbour algorithm, stating the Hamiltonian cycle produced when

i starting from vertex E

ii starting from vertex A. [8]

b State which is the better of these two upper bounds, and explain your reasoning. [1]

16 [Maximum mark: 9]

Paired bivariate data (x, y) is given in the following table:

x	0	1	2	3	4	5	6	7
y	0.3	0.7	1.3	2.7	5.3	11	21	43

a Giving your answer to 4 decimal places,

i find the Pearson product-moment correlation coefficient for the data (x, y)

ii find the Pearson product-moment correlation coefficient for the data $(x, \log y)$. [4]

b Comment on which of the two pairs of values in part (a) is more likely to be linked by a linear relationship. Give a reason for your answer. [1]

c State the equation of the linear regression line of $\log y$ on x, giving the coefficients correct to 4 decimal places. [2]

d Using your answer to part **c**, suggest a simple model which approximately connects x and y. Constant terms in your equation should be rounded to the nearest integer. [2]

Markscheme

1 **a** **i** $f\left(g\left(x\right)\right) = f\left(4x+5\right) = 2\left(4x+5\right)+3 = 8x+13$ M1 A1

ii $g\left(f\left(x\right)\right) = g\left(2x+3\right) = 4\left(2x+3\right)+5 = 8x+17$ M1 A1

[4 marks]

b $k\left(h\left(x\right)\right) = k\left(ax+b\right) = c\left(ax+b\right)+d = cax+cb+d$ A1

$h\left(k\left(x\right)\right) = h\left(cx+d\right) = a\left(cx+d\right)+b = acx+ad+b$ A1

Equality implies $cb+d = ad+b \Rightarrow ad-d = bc-b \Rightarrow$ M1 A1

$d(a-1) = b(c-1)$ AG

[4 marks]

[Total: 8 marks]

2 **a** $Y\left(1.05\right)^{10} = 1000 \Rightarrow Y = £613.91$ (2 d.p.) M1 A1

[2 marks]

b Require $(1.05^5 X - 1000)1.05^6 = 1000$ M1

$\Rightarrow 1.05^{11} X = \left(1+1.05^6\right)1000 \Rightarrow X = 1368.21$ (2 d.p.) A1 A1

[3 marks]

[Total: 5 marks]

3 $60^2 = 40^2 + 50^2 - 2\times 40\times 50\cos B$ M1 A1

$\Rightarrow 5 = 40\cos B \Rightarrow B = 82.819...$ A1

Bearing is $180 - 82.819 = 97.2^\circ$ (3 s.f.) M1 A1

[Total: 5 marks]

4 **a** (From graph on calculator) $0 \le t < 4.39$, $7.61 < t < 16.39$, $19.61 < t \le 24$ A2 A2 A2

[6 marks]

b $\dfrac{360}{30} = 12$ M1 A1

[2 marks]

[Total: 8 marks]

5 (By use of a lattice diagram or similar)

a $\dfrac{3}{16}$ A2

[2 marks]

b $\dfrac{3}{16}$ A2

[2 marks]

c $\dfrac{10}{16} = \dfrac{5}{8}$ A2

[2 marks]

[Total: 6 marks]

6 **a** Using a graph on the calculator, or substitution into the equation gives (M1)
£3390 A1

[2 marks]

b Identifying the maximum on the graph of P against x gives (M1)
£5950 by producing 5 cars A1 A1

[3 marks]

[Total: 5 marks]

7 $f(x) = \int 3x^2 + 8x + 7\,dx = x^3 + 4x^2 + 7x + c$ M1 A1 A1

Through $(1,2) \Rightarrow 1 + 4 + 7 + c = 2 \Rightarrow c = -10$ M1 A1

$f(x) = x^3 + 4x^2 + 7x - 10$ A1

[6 marks]
[Total: 6 marks]

8 a Let α be angle subtended by the sector at the centre of the circle.

Then $\pi r^2 \frac{\alpha}{360} = \frac{9\pi}{8}$ and $2\pi r \frac{\alpha}{360} = \frac{3\pi}{4}$ A1 A1

i Dividing first expression by second gives $\frac{r}{2} = \frac{3}{2} \Rightarrow r = 3\,\text{cm}$ M1 A1

ii Substituting $r = 3\,\text{cm}$ in first equation gives $\frac{\alpha}{360} = \frac{1}{8} \Rightarrow \alpha = 45^\circ$ M1 A1

[6 marks]

b $\frac{3\pi}{4} + 6 = 8.356... = 8$ cm (nearest cm) M1 A1

[2 marks]
[Total: 8 marks]

9 a As probabilities must add up to 1, $P(X = 0) = 0.39$ R1 A1

[2 marks]

b Fair game $\Rightarrow$ $E(X) = 0$ R1

$\Rightarrow -2 \times 0.2 - 1 \times 0.1 + 0 \times 0.39 + 1 \times 0.2 + 2 \times 0.1 + 0.01 \times a = 0$ M1 A1

$\Rightarrow a = 10$ A1

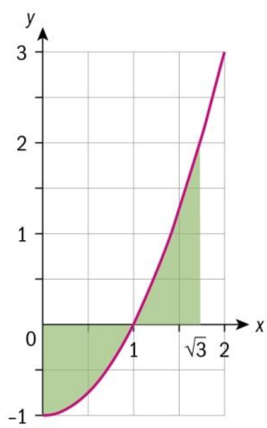
[4 marks]
[Total: 6 marks]

10 a $\int_0^a x^2 - 1\,dx = \left[\frac{x^3}{3} - x\right]_0^a = \frac{a^3}{3} - a$ M1 A1

$a\left(\frac{a^2}{3} - 1\right) = 0 \Rightarrow a^2 = 3 \Rightarrow a = \sqrt{3}$ M1 A1

[4 marks]

b



A1

$y(x)$ is below the x-axis for $0 < x < 1$, so $\int_0^1 y(x)\,\mathrm{d}x$ is negative

$y(x)$ is above the x-axis for $1 < x < \sqrt{3}$, so $\int_1^{\sqrt{3}} y(x)\,\mathrm{d}x$ is positive

These areas are equal in magnitude and opposite in sign, so they cancel out. R1

[2 marks]

[Total: 6 marks]

11 a $\frac{e^{-\mu}\mu^3}{3!} = \frac{e^{-\mu}\mu^4}{4!} \Rightarrow \mu = 4$ M1 A1 A1

[3 marks]

b $\sigma^2 = 4$ A1

[1 mark]

c It is bimodal at $X = 3$ and $X = 4$ R1 A1

[2 marks]

d $1 - P(X = 0) = 1 - e^{-4} = 0.982$ (3 s.f.) M1 A1

[2 marks]

[Total: 8 marks]

12 a $\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = 7\mathbf{I}$ A2

[2 marks]

b $A^2 - 6A = 7I \Rightarrow A \times \frac{1}{7}(A - 6I) = I \Rightarrow A^{-1} = \frac{1}{7}(A - 6I)$ M1 A1 A1

[3 marks]

[Total: 5 marks]

13 $|x + \mathrm{i}(y+2)| = |(-2y-1) + 2\mathrm{i}x|$ M1 A1

$\sqrt{x^2 + (y+2)^2} = \sqrt{(-2y-1)^2 + (2x)^2}$ M1 A1

$x^2 + y^2 + 4y + 4 = 4y^2 + 4y + 1 + 4x^2 \Rightarrow 3x^2 + 3y^2 = 3$ M1

$x^2 + y^2 = 1$, which is the equation of a circle A1 AG

Circle has centre $(0, 0)$ and radius 1 A1 A1

[8 marks]

[Total: 8 marks]

14 Let the cone have radius r and height h.

Then $V = \frac{1}{3}\pi r^2 h$ A1

Given $\frac{\mathrm{d}V}{\mathrm{d}t} = 2$

Need $\frac{\mathrm{d}h}{\mathrm{d}t}$

Use of chain rule $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}\left(= \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h}\right)$ M1

Sloped surface, vertical and the radius form a right-angled triangle, so

$\frac{r}{h} = \tan\frac{\pi}{3} = \sqrt{3} \Rightarrow r^2 = 3h^2$ M1 A1

Substituting $r^2 = 3h^2$ into $V = \frac{1}{3}\pi r^2 h$ gives $V = \pi h^3$ A1

$\frac{dV}{dh} = 3\pi h^2$ A1

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = 2 \times \frac{1}{3\pi h^2}$ M1

At $h = 5$ $\frac{dh}{dt} = \frac{2}{75\pi} = 0.00849\,\text{m s}^{-1}$ A1

[Total: 8 marks]

15 a i EABCDE; $1+3+5+7+14=30$ M1 A2 A1

ii AEBCDA; $1+9+5+7+11=33$ M1 A2 A1

[8 marks]

b Starting at E is better as the distance travelled is shorter. R1

[1 mark]

[Total: 9 marks]

16 a i $r = 0.8497$ (4 d.p.) A2

ii $r = 0.9997$ (4 d.p.) A2

[4 marks]

b $(x, \log y)$ is more likely to be lined by a linear relationship p as it has a Pearson product-moment correlation coefficient almost equal to 1, indicating an almost perfect, positive linear correlation. R1

[1 mark]

c $\log y = 0.3042x - 0.4912$ A1 A1

[2 marks]

d $y = 10^{0.3042\ldots x - 0.4912\ldots}$ $y = \frac{\left(10^{0.3042\ldots}\right)^x}{10^{0.4912\ldots}}$ M1

$y \simeq \frac{2^x}{3}$ A1

[2 marks]

[Total: 9 marks]

Paper 2

Time allowed: 2 hours

Maximum number of marks: 110 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphical display calculator for this paper.

1 [Maximum mark: 12]

In a flat country there are two ancient, straight, Roman roads. Relative to a co-ordinate system, the two roads have equations $y = 2x + 3$ and $7y + 4x = 75$. Units are in kilometres.

a The roads cross at a village at point A. Find the coordinates of point A. [2]

b Determine, with a reason, whether or not these two roads are perpendicular. [3]

c Point B lies on $y = 2x + 3$ and has $x = -2$. Point C lies on $7y + 4x = 75$ and has $y = 5$. Find the co-ordinates of (i) point B, (ii) point C. [2]

d Point M is the mid-point of the line section BC. Find the coordinates of point M. [2]

e A drone is to be flown from point B to point C at ground level. Calculate the distance it will travel. [3]

2 [Maximum mark: 16]

Two hundred married couples, of different age groups, were asked their preferences about a celebratory meal. Their responses are given in the table below.

Meal\Age	Young	Middle-aged	Elderly
Take-away	28	20	5
Eat out	20	35	22
Cook in	16	26	28

One of the couples is selected at random.

a Find the probability that they

i are elderly **ii** prefer take-away **iii** are middle-aged and prefer to eat out

iv are elderly, given that they prefer take-away

v prefer take-away, given that they are elderly. [5]

It is thought that meal preference and age group are dependent on one another.

b Devise and carry out a test, to test this hypothesis at the 5% level.

You should:

- state the name of the test being used
- state the hypotheses
- under the null hypothesis, give a table of expected frequencies in a similar format to that above and comment on these values with regard to the validity of the test
- state the number of degrees of freedom
- state the p-value.

With a reason, state the conclusion of the test. Give your answer in the context of the question. [11]

3 [Maximum mark: 12]

Paired bivariate data (x, y) is collected from 11 students, where x is their time to swim 100 m (measured in seconds) and y is their time to run 200 m (also measured in seconds). The data is given in the following table:

x	100	81	120	104	180	200	152	102	94	131	142
y	40	35	44	39	51	60	48	40	37	43	47

a Calculate the Pearson product moment correlation coefficient (r) for this data, and state what this value of r implies about the relationship between the swimming and running times. [5]

b **i** Calculate the equation of the linear regression line of y on x.

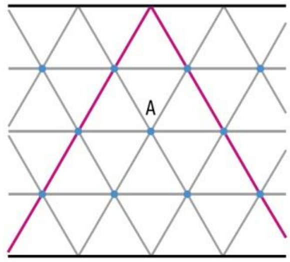
ii Write down the mean point $(\bar{x}, \bar{y})$ that the linear regression line of y on x must pass through.

iii A twelfth student had a swimming time of 110 seconds. Estimate their running time. [5]

c State two reasons why the equation found in part **b i** should not be used to estimate the swimming time of a student with a running time of 23 seconds. [2]

4 [Maximum mark: 14]

In a very large country, mobile phone transmitter masts are placed at the vertices of an equilateral triangle grid, as shown in the diagram:



The sides of the equilateral triangles are each 20 km long.

a **i** Copy the diagram and sketch in the lines to show the Voronoi diagram cell about the mast labelled *A.*

ii Describe the shape of this cell. [5]

b Find the area of this cell, which will be the area of the ground that is controlled by mast *A*. [4]

c Find the furthest distance that a person could be from a mast. [3]

d Find the **exact** value of the ratio

area of equilateral triangle formed by three adjacent masts : area of a cell found in part **b**. [2]

5 [Maximum mark: 10]

A famous painting is being restored. A coordinate system is projected onto the painting and a brush stroke, in the shape of a curve, passes through the following points:

$(1,2),(2,2.5),(3,4),(4,5.5),(5,5.75),(6,6)$

The restorer wants to find an approximation for the equation of the curve. He wants to know which of the following best approximates the curve:

i quadratic regression

ii cubic regression

iii logarithmic regression of the form $y = a + b\ln x$

iv power regression of the form $y = ab^x$.

a Find the value of R^2 for each of these 4 regression curves and hence state, with a reason, which type of regression best fits this curve. [6]

b For the most suitable type of regression identified in part **a**, write down the equation of the regression curve of *y* on *x*. Use this equation to estimate the value of *y* when $x=3.5$. [3]

c Explain why it would not be helpful to use this regression line to calculate the value of *y* when $x=10$. [1]

6 [Maximum mark: 14]

An undirected graph has the following adjacency matrix:

	A	B	C	D
A	0	1	0	1
B	1	0	2	0
C	0	2	0	1
D	1	0	1	0

a Draw this graph in a planar fashion. [2]

b State, with a reason, whether or not this graph is simple. [2]

c **i** State whether or not this graph has an Eulerian circuit. If so, give an example of such a circuit, and if not, give a reason why not.

ii State whether or not this graph has an Eulerian open trail. If so, give an example of such a circuit, and if not, give a reason why not. [4]

d Find the number of walks of length

i four, from vertex A to vertex A

ii three or less, from vertex A to vertex B. [6]

7 [Maximum mark: 18]

The number of lemurs, P, on a small island off Madagascar is increasing. Initially, $P = 100$. The rate of change in the population is given by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.5P\left(1 - \frac{P}{1000}\right)$$

where t is time, measured in years.

a Show that $\dfrac{1}{P\left(1 - \dfrac{P}{1000}\right)} = \dfrac{1}{P} + \dfrac{\dfrac{1}{1000}}{\left(1 - \dfrac{P}{1000}\right)}$. [1]

b Solve the differential equation to show that P satisfies the logistic model equation

$$P = \frac{1000}{1 + 9e^{-0.5t}}\ .$$ [12]

c Find the limit that P tends to as t tends to infinity. [1]

d Find the value of P when $t = 10$. [2]

e Find the value of t for which $P = 500$. [2]

8 [Maximum mark: 14]

a State the Central Limit Theorem for a random variable X with mean μ and variance σ^2. [4]

Let the random variable X be the number of days that it takes a patient to fully recover after a hip replacement operation. It is known that the standard deviation of X is 5 years.

b A random sample of 100 patients were taken and it was found that $\bar{X} = 202$.

i Find the 90% confidence interval for μ, giving the values to one decimal place.

ii By considering 100 such samples being taken and the intervals calculated, state what is meant by a 90% confidence interval for μ. [3]

c The leading surgeon for these operations believes that $\mu = 200$. Another random sample of 100 patients is to be taken to test this claim, against the alternative that $\mu > 200$. The test will be carried out at the 5% level. Find the critical region for $\bar{X}$, giving the value to one decimal place. [2]

d **i** If the test mentioned in part **c** is to be carried out, write down the probability of a Type I error.

ii If the actual value of μ is 201 and the test mentioned in part **c** is to be carried out, find the probability of a Type II error, giving your answer to two decimal places. [5]

Markscheme

1 a Solving $y = 2x + 3$ and $7y + 4x = 75$ gives $A = (3, 9)$ M1 A1

[2 marks]

b $y = 2x + 3$ has gradient of 2. $7y + 4x = 75$ has gradient of $-\frac{4}{7}$ A1 A1

$-\frac{4}{7} \neq -\frac{1}{2}$ so not perpendicular. R1

[3 marks]

c i $B = (-2, -1)$ **ii** $C = (10, 5)$ A1 A1

[2 marks]

d $M = \left(\frac{-2+10}{2}, \frac{-1+5}{2}\right) = (4, 2)$ M1 A1

[2 marks]

e $\sqrt{(10 - -2)^2 + (5 - -1)^2} = \sqrt{12^2 + 6^2} = 13.4\,\text{km}\ (3\,\text{s.f.})$ M1 A2

[3 marks]

[Total:12 marks]

2 a i $\frac{55}{200} = \frac{11}{40}$ **ii** $\frac{53}{200}$ **iii** $\frac{35}{200} = \frac{7}{40}$ A1 A1 A1

iv $\frac{5}{53}$ **v** $\frac{5}{55} = \frac{1}{11}$ A1 A1

[5 marks]

b ψ^2 test for independence A1

H_0 : meal preference and age group are independent A1

H_1 : meal preference and age group are dependent A1

Meal\Age	Young	Middle aged	Elderly
Take-away	16.96	21.465	14.575
Eat out	24.64	31.185	21.175
Cook in	22.40	28.35	19.25

A2

All these values are >5, making the test valid. R1

4 degrees of freedom A1

p-value$= 3.24 \times 10^{-4} < 0.05$ A2

So we reject H_0 and conclude that meal preference and age group are dependent.

R1 A1

[11 marks]

[Total: 16 marks]

3 a $r = 0.979$ (3 s.f.) A2

Strong, positive, linear correlation A1 A1 A1

[5 marks]

b i $y = 0.187x + 20.1$ A1 A1

ii (128,44) A1

iii $y = 0.187(110) + 20.1 = 40.7$ (3 s.f.) M1 A1

[5 marks]

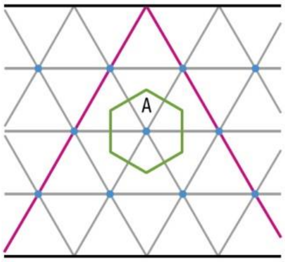
c The line of x on y should be used instead when estimating a swimming time from a running time. R1

Using the line to estimate a swimming time when the running time is 23 seconds would be extrapolation a long way away from the given data. R1

[5 marks]

[Total: 12 marks]

4 a i



A3

ii A regular hexagon A1 A1

[5 marks]

b Cell consists of 6 smaller equilateral triangles each of height 10 km R1

For each small triangle, $\frac{1}{2}$ base length $= 10\tan 30 = \frac{10}{\sqrt{3}}$ A1

Total area of regular hexagon $= 6 \times 10 \times \frac{10}{\sqrt{3}} = 200\sqrt{3} = 346\,\text{km}^2$ (3 s.f.) M1 A1

[4 marks]

c Furthest distance will be length of hypotenuse of right-angled triangle (half of one smaller equilateral triangle) R1

$= \frac{10}{\cos 30} = \frac{20}{\sqrt{3}} = 11.5\,\text{km}$ (3 s.f.) M1 A1

[3 marks]

d Original equilateral triangle has area $= \frac{1}{2} \times 20 \times 20 \sin 60 = 100\sqrt{3}$ M1

Area of cell found in (b) is $200\sqrt{3}$, so ratio is $100\sqrt{3} : 200\sqrt{3}$, which is 1:2 A1

[2 marks]

5 a i 0.958 **ii** 0.986

iii 0.922 **iv** 0.946 A1 A1 A1 A1

Cubic regression is best A1

The value of R^2 is largest for cubic regression, indicating that the data is best linked by a cubic curve. R1

[6 marks]

b $y = -0.0810x^3 + 0.739x^2 - 0.894x + 2.17$ A2

$y(3.5) = 4.62$ (3 s.f.) A1

[3 marks]

c This is extrapolation well away from the data and thus the regression line may not fit the data here; the brush stroke might not even extend this far. R1

[1 mark]

[Total: 10 marks]

6 a

A2

[2 marks]

b No; there are multiple edges A1 R1

[2 marks]

c **i**	No; there are 2 vertices of odd degree	A1	R1
ii	Yes; e.g. BADCBC	A1	A1
		[4 marks]	
d **i**	Appropriate entry in M^4; 13 walks	M1	A2
ii	Appropriate entry in M^3+M^2+M; 9 walks	M1	A2
		[6 marks]	
		[Total: 14 marks]	

7 **a** RHS $= \frac{1}{P} + \frac{\frac{1}{1000}}{\left(1-\frac{P}{1000}\right)} = \frac{\left(1-\frac{P}{1000}\right)+\frac{P}{1000}}{P\left(1-\frac{P}{1000}\right)} = \frac{1}{P\left(1-\frac{P}{1000}\right)} =$ LHS — M1 AG

[1 mark]

b $\int \frac{1}{P\left(1-\frac{P}{1000}\right)}\,\mathrm{d}P = \int 0.5\,\mathrm{d}t$ — M1 A1

$\int \frac{1}{P} + \frac{\frac{1}{1000}}{\left(1-\frac{P}{1000}\right)}\,\mathrm{d}p = 0.5t + c$ — M1 A1

$\ln P - \ln\left(1-\frac{P}{1000}\right) = 0.5t + c$ — A1

$\ln\left(\frac{P}{\left(1-\frac{P}{1000}\right)}\right) = 0.5t + c \Rightarrow \frac{P}{\left(1-\frac{P}{1000}\right)} = e^{0.5t+c} = Ae^{0.5t}$ — M1 A1 A1

$t = 0, P = 100$ gives $A = \frac{100}{1-\frac{1}{10}} = \frac{1000}{9}$ — M1 A1

$\frac{P}{\left(1-\frac{P}{1000}\right)} = \frac{1000}{9}e^{0.5t} \Rightarrow 9P = (1000-P)e^{0.5t}$

$P\left(9+e^{0.5t}\right) = 1000e^{0.5t} \Rightarrow P = \frac{1000e^{0.5t}}{9+e^{0.5t}}$ — M1 A1

$\Rightarrow P = \frac{1000}{1+9e^{-0.5t}}$ — AG

[12 marks]

c As $t \to \infty, e^{-0.5t} \to 0, P \to 1000$ — R1

[1 mark]

d $t = 10 \Rightarrow P = \frac{1000}{1+9e^{-5}} = 943\ (3\,\text{s.f.})$ — M1 A1

[2 marks]

e $500 = \frac{1000}{1+9e^{-0.5t}} \Rightarrow 2 = 1 + 9e^{-0.5t} \Rightarrow e^{-0.5t} = \frac{1}{9} \Rightarrow e^{0.5t} = 9 \Rightarrow 0.5t = \ln 9$

$t = 2\ln 9 = 4.39$ years (3 s.f.) — M1 A1

[2 marks]

[Total: 18 marks]

8 **a** If an independent sample of size n is taken, then for n sufficiently large, — A1

the sample mean, $\bar{X}$, — A1

will be approximately normally distributed with mean of μ — A1

and variance of $\frac{\sigma^2}{n}$. — A1

[4 marks]

b **i** $[201.2, 202.8]$ A1 A1

ii Of the 100 intervals calculated, the expected number of them that would contain μ would be 90. A1

[3 marks]

c Under H_0, $\bar{X} \sim N\left(200, \frac{25}{100}\right)$ and $P\left(\bar{X} < A\right) = 0.95 \Rightarrow A = 200.82...$

So critical region is $\bar{X} > 200.8$. M1 A1

[2 marks]

d **i** (Probability of Type I error is the level of the test) 0.05 A1

ii Probability of a type II error equals the probability of accepting H_0 when it is false A1

$= P\left(\bar{X} < 200.8 \middle| \bar{X} \sim N\left(201, \frac{25}{100}\right)\right) = 0.3445...$ so 0.34 (2 d.p.) M1 A1 A1

[5 marks]

[Total: 14 marks]

Paper 3

Time allowed: 1 hour

Maximum number of marks: 60 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphic display calculator for this paper.

1 [Maximum mark: 20]

In this question you will investigate approximations for the volume of a solid of revolution.

The line segment joining the points $(0, y_0)$ and (h, y_1) is rotated through 2π radians about the x-axis to form a solid of revolution.

a Show that the volume of the solid of revolution is given by $\frac{\pi h}{3}\left(y_0^2 + y_0 y_1 + y_1^2\right)$. [7]

b A curve with equation $y = f(x)$ is rotated through 2π radians about the x-axis to form a solid of revolution. Use your answer to part **a** to show that the equivalent formula to the Trapezium rule, but for a solid of revolution, is that the volume of the solid obtained is approximately

$$\frac{\pi h}{3}\left(y_0^2 + 2(y_1^2 + y_2^2 + \dots y_{n-1}^2) + y_n^2 + y_0 y_1 + y_1 y_2 + \dots y_{n-1} y_n\right)$$

where, with the same notation as the trapezium rule, there are n strips each of width h, with the ordinates being $y_0, y_1, \dots y_n$. [3]

c A piece of sculpture is made by rotating a curve, which passes through points $(0,1), (1,2), (2,1), (3,0)$ relative to a co-ordinate system, through 2π radians about the x-axis to form a solid of revolution. Use the result in part **b** with $h = 1$ to find an approximation to the volume of the sculpture. [2]

You will now consider an alternative method to approximate the volume of the sculpture in part **c**.

d Use quadratic regression to find the equation of a best fit quadratic curve through the points given in part **c**. State the R^2 value for this regression, and comment on this value. [4]

e Use numerical integration to find the volume of the solid of revolution formed when a portion of the quadratic curve you found in part (d), from $x = 0$ to $x = 3$, is rotated through 2π radians about the x-axis. [3]

f Explain why the approximation found in part (c) is smaller than the approximation found in part **e**. [1]

2 [Maximum mark: 40]

In this question, you will work with 2nd order differential equations, such as those used to model simple harmonic motion.

An object is moving in a straight line under the effect of a spring. Its motion satisfies the differential equation

$$\frac{d^2x}{dt^2} + 9x = 0$$

where x is displacement in centimetres and t is time in seconds.

a Show that $x = A\sin 3t + B\cos 3t$ is a solution of this differential equation for any constants A and B. [3]

b Letting $\frac{dx}{dt} = y$ we can let the differential equation be written as the coupled differential equations $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -9x$. Solve these differential equations using the eigenvalue method to show that $x = De^{3it} + Ee^{-3it}$. [11]

c Using the relationship $e^{i\theta} = \cos\theta + i\sin\theta$, show that the answer obtained in part **b** is equivalent to that obtained in part **a**. [2]

d Given that when $t = 0$ we have $x = 0$ and $\frac{dx}{dt} = 3$, find the particular solution to the equation from part **a** which fits these initial conditions. [5]

We will now consider another object moving in a straight line which satisfies the differential equation

$$\frac{d^2z}{dt^2} - 9z = 0$$

where z is displacement in centimetres and t is time in seconds.

e Use the eigenvalue method of part (b) to solve this differential equation. [11]

We will now introduce some new functions:

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \ \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} .$$

f Find and simplify the derivatives of $\cosh\theta$ and $\sinh\theta$. [2]

g Show that the answer obtained in part (e) can be written as

$z = a\sinh 3t + b\cosh 3t$, where a and b are constants. [2]

h Given that when $t = 0$ we have $z = 0$ and $\frac{dz}{dt} = 3$, find the particular solution to the equation from part (g) which fits these initial conditions. [4]

Markscheme

1 **a** Gradient of line is $\frac{y_1 - y_0}{h}$. Equation of line is $y = \frac{y_1 - y_0}{h}x + y_0$ M1 A1

Volume is $\pi\int_0^h \left(\frac{y_1 - y_0}{h}x + y_0\right)^2 \mathrm{d}x = \pi\left[\frac{1}{3}\frac{h}{y_1 - y_0}\left(\frac{y_1 - y_0}{h}x + y_0\right)^3\right]_0^h$ M1 A1

$= \frac{\pi}{3}\frac{h}{y_1 - y_0}\left(y_1^3 - y_0^3\right) = \frac{\pi h}{3}\left(y_1^2 + y_1 y_0 + y_0^2\right)$ M1 A1 M1

$= \frac{\pi h}{3}\left(y_0^2 + y_0 y_1 + y_1^2\right)$ AG

[7 marks]

b In part **a**, we saw that when one such trapezium (between ordinates y_0 and y_1) is rotated about the x-axis, the solid obtained had volume $= \frac{\pi h}{3}\left(y_0^2 + y_0 y_1 + y_1^2\right)$. Now consider n trapeziums being rotated about the x-axis. The total volume of the solid obtained will be the sum of n similar expressions to that obtained in part **a**. (R1)

$\frac{\pi h}{3}\left(y_0^2 + y_0 y_1 + y_1^2\right) + \frac{\pi h}{3}\left(y_1^2 + y_1 y_2 + y_2^2\right) + \ldots + \frac{\pi h}{3}\left(y_{n-1}^2 + y_{n-1} y_n + y_n^2\right)$ M1 A1

$\frac{\pi h}{3}\left(y_0^2 + 2(y_1^2 + y_2^2 + \ldots y_{n-1}^2) + y_n^2 + y_0 y_1 + y_1 y_2 + \ldots y_{n-1} y_n\right)$ AG

[3 marks]

c Volume $\simeq \frac{\pi}{3}\left(1^2 + 2\left(2^2 + 1^2\right) + 0^2 + 2 + 2 + 0\right) = 5\pi = 15.7$ (3 s.f.) M1 A1

[2 marks]

d $y = -0.5x^2 + 1.1x + 1.1$ A2

$R^2 = 0.9$ which indicates strong correlation, so this regression curve is a good fit. A1 R1

[4 marks]

e $\pi\int_0^3 \left(-0.5x^2 + 1.1x + 1.1\right)^2 \mathrm{d}x = 16.9$ (3 s.f.) M1 A2

[3 marks]

f Volume found in part **c** is smaller than that found in part **e** since the quadratic curve is concave down, and thus the trapeziums will lie entirely below the curve. R1

[1 mark]

[Total: 20 marks]

2 **a** $x = A\sin 3t + B\cos 3t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 3A\cos 3t - 3B\sin 3t$ M1

$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9A\sin 3t - 9B\cos 3t = -9x$ A1 A1

Hence, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 9x = 0$ AG

[3 marks]

b $\det\begin{pmatrix} -\lambda & 1 \\ -9 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3\mathrm{i}$ are the eigenvalues M1 A1 A1

$\lambda = 3\mathrm{i}$ $\begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 3\mathrm{i}\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow q = 3\mathrm{i}p$, an eigenvector is $\begin{pmatrix} 1 \\ 3\mathrm{i} \end{pmatrix}$ M1 A1 A1

$\lambda = -3\mathrm{i}$ $\begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -3\mathrm{i}\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow q = -3\mathrm{i}p$, an eigenvector is $\begin{pmatrix} 1 \\ -3\mathrm{i} \end{pmatrix}$ M1 A1 A1

$\begin{pmatrix} x \\ y \end{pmatrix} = De^{3\mathrm{i}t}\begin{pmatrix} 1 \\ 3\mathrm{i} \end{pmatrix} + Ee^{-3\mathrm{i}t}\begin{pmatrix} 1 \\ -3\mathrm{i} \end{pmatrix}$ M1 A1

$\Rightarrow x = De^{3\mathrm{i}t} + Ee^{-3\mathrm{i}t}$ AG

[11 marks]

c $x = D(\cos 3t + \mathrm{i}\sin 3t) + E(\cos(-3t) + \mathrm{i}\sin(-3t))$ M1

$= (D+E)\cos 3t + \mathrm{i}(D-E)\sin 3t$ A1

$x = A\sin 3t + B\cos 3t$ as before [2 marks]

d Using initial conditions, $B = 0$ M1 A1

$\frac{dx}{dt} = 3A\cos 3t = 3$ when $t = 0 \Rightarrow A = 1$ M1 A1

$x = \sin 3t$ A1

[5 marks]

e Let $w = \frac{\mathrm{d}z}{\mathrm{d}t}$, we have the coupled equations $\frac{\mathrm{d}z}{\mathrm{d}t} = w$, $\frac{\mathrm{d}w}{\mathrm{d}t} = 9z$

$\det\begin{pmatrix} -\lambda & 1 \\ 9 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$ are the eigenvalues M1 A1 A1

$\lambda = 3$ $\begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 3\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow q = 3p$, an eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ M1 A1 A1

$\lambda = -3$ $\begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -3\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow q = -3p$, an eigenvector is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ M1 A1 A1

$\begin{pmatrix} z \\ w \end{pmatrix} = Fe^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + Ge^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \Rightarrow z = Fe^{3t} + Ge^{-3t}$ M1 A1

[11 marks]

f $\frac{\mathrm{d}(\cosh\theta)}{\mathrm{d}\theta} = \frac{e^{\theta} - e^{-\theta}}{2} = \sinh\theta$, $\frac{\mathrm{d}(\sinh\theta)}{\mathrm{d}\theta} = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta$ A1 A1

[2 marks]

g $z = F(\cosh 3t + \sinh 3t) + G(\cosh 3t - \sinh 3t) = (F-G)\sinh 3t + (F+G)\cosh 3t$ M1 A1

So $z = a\sinh 3t + b\cosh 3t$ AG

[2 marks]

h Using initial conditions $b = 0$ A1

$\frac{\mathrm{d}z}{\mathrm{d}t} = 3a\cosh 3t = 3$ when $t = 0 \Rightarrow a = 1$ M1 A1

$z = \sinh 3t$ A1

[4 marks]

[Total: 40 marks]